UCID - 16750



LAWRENCE LIVERMORE LABORATORY
*University of California/Livermore, California*

# QUALITATIVE ASPECTS OF PHASE MODULATION IN SELF-INDUCED TRANSPARENCY

R. C. Harney

MASTER

March 15, 1974

## DISCLAIMER

## DISCLAIMER

QUALITATIVE ASPECTS OF PHASE MODULATION
IN SELF-INDUCED TRANSPARENCY*

R. C. Harney

University of California Lawrence Livermore Laboratory
Livermore, CA 94550

ABSTRACT

It is shown that retention of terms previously considered negligible in the equations of self-induced transparency invariably leads to phase-modulated solutions. The presence of this phase modulation will modify the propagation characteristics of the laser pulse and may cause the $2\pi$ hyperbolic secant pulse to become unstable. It is also shown that the initial magnitude of the phase modulation is much less than the pulse bandwidth and that the area theorem of McCall and Hahn is approximately valid for propagation through distances less than $\omega_0 T_2^*/\alpha$. In this context, it may be concluded that the original results of McCall and Hahn are sufficient to describe most situations of experimental interest.

*This work was performed under the auspices of the U.S. ERDA.

The propagation of coherent light pulses through a resonant medium composed of two-level atoms embedded in a lossless host is governed by the electromagnetic wave equation and the electric Bloch equations. These coupled nonlinear differential equations relate the electric field of the pulse to the medium's nonlinear macroscopic polarization, and vice versa. We consider circularly-polarized fields of the form

$$E_+(z,t) = \varepsilon(z,t)\, e^{i[\omega_o t - k_o z - \phi(z,t)]} \tag{1}$$

and polarizations of the form

$$P_+(z,t) = [P_1(z,t) + iP_2(z,t)]e^{i[\omega_o t - k_o z - \phi(z,t)]} . \tag{2}$$

Here $\varepsilon$ (the envelope function) stays real as the pulse propagates. Substitution of the appropriate derivatives of Eqs. (1) and (2) into the wave equation,

$$\frac{\partial^2 E_+}{\partial z^2} - \left(\frac{n_o}{c}\right)^2 \frac{\partial^2 E_+}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P_+}{\partial t^2} , \tag{3}$$

allows it to be decomposed into the two equations,

$$\begin{aligned}\left[\frac{\partial^2 \varepsilon}{\partial z^2} - \left(k_o + \frac{\partial\phi}{\partial z}\right)^2 \varepsilon\right] - \left(\frac{n_o}{c}\right)^2 \left[\frac{\partial^2 \varepsilon}{\partial t^2} - \left(\omega_o - \frac{\partial\phi}{\partial t}\right)^2 \varepsilon\right] = \\ \frac{4\pi}{c^2}\left[\frac{\partial^2 P_1}{\partial t^2} + P_2 \frac{\partial^2 \phi}{\partial t^2} - 2\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\frac{\partial P_2}{\partial t} - \left(\omega_o - \frac{\partial\phi}{\partial t}\right)^2 P_1\right]\end{aligned} \tag{4}$$

and

$$\left[2\left(k_o + \frac{\partial\phi}{\partial z}\right)\frac{\partial\varepsilon}{\partial z} + \varepsilon\frac{\partial^2\phi}{\partial z^2}\right] - \left(\frac{n_o}{c}\right)^2\left[\varepsilon\frac{\partial^2\phi}{\partial t^2} - 2\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\frac{\partial\varepsilon}{\partial t}\right] =$$

$$\frac{4\pi}{c^2}\left[-\frac{\partial^2 P_2}{\partial t^2} + P_1\frac{\partial^2\phi}{\partial t^2} - 2\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\frac{\partial P_1}{\partial t} + \left(\omega_o - \frac{\partial\phi}{\partial t}\right)^2 P_2\right]. \tag{5}$$

$n_o$ in Eqs. (3)-(5) is the refractive index of the medium and accounts for the linear polarization induced in the host.

For a pseudo-polarization vector of the form $\vec{P} = u\hat{u} + v\hat{v} - w\hat{w}$,[1] the electric Bloch equations describing the behavior of the two-level atoms (in a coordinate frame rotating with angular frequency $\omega_o - \frac{\partial\phi}{\partial t}$ ) may be written

$$\frac{du}{dt} = (\Delta\omega + \frac{\partial\phi}{\partial t})v - \frac{u}{T_2'} \quad , \tag{6}$$

$$\frac{dv}{dt} = -(\Delta\omega + \frac{\partial\phi}{\partial t})u - \kappa\varepsilon w - \frac{v}{T_2'} \quad , \tag{7}$$

and

$$\frac{dw}{dt} = \kappa\varepsilon v - \frac{w-w_o}{T_1} \quad . \tag{8}$$

$T_1$ is the longitudinal relaxation time, $T_2'$ is the homogeneous contribution to the transverse relaxation time, and $\Delta\omega = \omega-\omega_o$ represents the difference between the laser carrier frequency $\omega_o$ and the natural resonance frequency $\omega$ of the atomic system. $\kappa = \frac{2p}{\hbar}$ is the gyroelectric ratio, where p is the dipole matrix element connecting the two energy levels of the atomic system. For an inhomogeneously-broadened medium, u and v are related to the real and imaginary components of the nonlinear polarization by

$$P_1(z,t) = Np\int_{-\infty}^{\infty} g(\Delta\omega)u(\Delta\omega,z,t)\, d\Delta\omega \tag{9}$$

and

$$P_2(z,t) = Np\int_{-\infty}^{\infty} g(\Delta\omega)v(\Delta\omega,z,t)\, d\Delta\omega \qquad (10)$$

where $g(\Delta\omega)$ is the inhomogeneous lineshape function and N is the number of two-level systems per unit volume.

Equations (4)-(8) have been investigated both numerically and analytically under different approximations by a number of authors.[3-13] For example, McCall and Hahn[3-4] assumed that Eqs. (4)-(8) could be approximated in the limit of $T_1$, $T_2' \gg \tau_p$ by the first order set of equations

$$\frac{\partial\varepsilon}{\partial z} + \frac{n_o}{c}\frac{\partial\varepsilon}{\partial t} = \frac{2\pi\omega_o}{n_o c} P_2 \ , \qquad (11)$$

$$\frac{du}{dt} = \Delta\omega v \ , \qquad (12)$$

$$\frac{dv}{dt} = -\Delta\omega u - \kappa\varepsilon w \ , \qquad (13)$$

and

$$\frac{dw}{dt} = \kappa\varepsilon v \ . \qquad (14)$$

They assumed a symmetric lineshape centered at $\omega_o$ and looked for solutions in the absence of phase modulation. Within the framework of Eqs. (11)-(14), McCall and Hahn discovered the self-induced transparency effect, in which a pulse of the proper shape propagates through the medium without distortion. Their steady state solutions are

$$u(\Delta\omega,z,t) = \frac{2\tau\Delta\omega \sin\delta/2}{1 + (\tau\Delta\omega)^2} \ , \qquad (15)$$

$$v(\Delta\omega,z,t) = \frac{\sin\delta}{1 + (\tau\Delta\omega)^2} \ , \qquad (16)$$

and

$$\varepsilon(z,t) = \frac{2}{\kappa\tau} \text{ sech } \beta \quad , \tag{17}$$

where

$$\beta = \frac{1}{\tau} \left(t - \frac{z}{V}\right) \tag{18}$$

is the retarded time divided by a characteristic pulse duration $\tau$ and

$$\delta = 4 \tan^{-1} (e^{\beta}) \ . \tag{19}$$

In addition they discovered that any un-phase-modulated input pulse of sufficient intensity would evolve toward the steady state solutions as it propagates through the medium. The proof of this behavior is embodied in the "area theorem".[4]

McCall and Hahn justified their neglect of phase modulation by substituting their results into the phase equation

$$\frac{\partial\phi}{\partial z} + \frac{n_o}{c} \frac{\partial\phi}{\partial t} = \frac{2\pi\omega_o}{n_o c\varepsilon} P_1 \ . \tag{20}$$

Eqs. (9) and (15) show that for a resonant symmetric lineshape, $P_1 \equiv 0$ in their solution and thus there is no phase driving term in Eq. (20). As a result McCall and Hahn concluded that there is no phase modulation in self-induced transparency with single $2\pi$ pulses. The same conclusion was reached by Eberly and Matulic[8,9] after considering Eq. (20) simultaneously with Eqs. (11)-(14), modified by substitution of $\Delta\omega - \frac{\partial\phi}{\partial t}$ for $\Delta\omega$.

Diels and Hahn[10] studied equations similar to those of Matulic and Eberly for the case of a symmetric lineshape which is not resonant with the carrier frequency. In this case, $P_1 \neq 0$ and some phase modulation is to be expected. Thus, it is not surprising that Diels and Hahn observed substantial frequency pulling and frequency pushing effects in their computer simulation. Similar effects should also be observed in the case of an asymmetric lineshape, resonant or otherwise. Recently, Lee[11] solved the equations of self-induced transparency without making the usual slowly-varying envelope and phase approximations. In his approximation, valid for extremely short pulses, he obtained analytic phase-modulated solutions. However, in all cases of practical interest, Lee's solutions are experimentally indistinguishable from the McCall and Hahn solutions.[14] In the following paragraphs we re-examine the question of phase modulation in pure self-induced transparency.

As Eqs. (4) and (5) are too complicated to yield readily to analysis, they must be simplified. If we restrict our consideration to pulses whose amplitudes and instantaneous frequencies do not change appreciably over one optical cycle, we may apply the slowly-varying envelope and phase approximations (SVEA). Consult the appendix for the mathematics of these approximations. Upon application of these approximations, Eqs. (4) and (5) reduce to

$$\left[\frac{\partial^2 \varepsilon}{\partial z^2} - \left(\frac{n_o}{c}\right)^2 \frac{\partial^2 \varepsilon}{\partial t^2}\right] - 2k_o\varepsilon\left[\frac{\partial \phi}{\partial z} + \frac{n_o}{c}\frac{\partial \phi}{\partial t}\right] = \frac{4\pi}{c^2}\left[-\omega_o{}^2 P_1 - 2\omega_o \frac{\partial P_2}{\partial t} + \frac{\partial^2 \phi}{\partial t^2} P_2\right] \tag{21}$$

and

$$\varepsilon\left[\frac{\partial^2 \phi}{\partial z^2} - \left(\frac{n_o}{c}\right)^2 \frac{\partial^2 \phi}{\partial t^2}\right] + 2k_o\left[\frac{\partial \varepsilon}{\partial z} + \frac{n_o}{c}\frac{\partial \varepsilon}{\partial t}\right] = \frac{4\pi}{c^2}\left[\omega_o{}^2 P_2 - 2\omega_o \frac{\partial P_1}{\partial t} + \frac{\partial^2 \phi}{\partial t^2} P_1\right] . \tag{22}$$

These equations differ from those usually quoted as resulting from the application of the SVEA. We believe the reason for this discrepancy lies in a non-rigorous application of this approximation by most previous authors.

Mathematical rigor requires that a term be neglected only if it is always small compared to another term in the same equation. Thus, even though $\omega_o \frac{\partial P_2}{\partial t}$ may be neglected compared to $\omega_o^2 P_2$, it may not be neglected when compared to $\omega_o^2 P_1$. It is also important that approximations only be applied after all rigorous mathematical simplifications, such as cancellation of terms, have been made in the equations. Subject to these guidelines, Eqs. (21) and (22) follow from Eqs. (4) and (5) upon application of the SVEA.

A further simplification of Eqs. (21) and (22) can be made by assuming that all second-derivative terms are negligible. This leads to the first-order equations

$$\frac{\partial \varepsilon}{\partial z} + \frac{n_o}{c}\frac{\partial \varepsilon}{\partial t} = \frac{2\pi}{n_o c}\left[\omega_o P_2 - 2\frac{\partial P_1}{\partial t}\right] \tag{23}$$

and

$$\frac{\partial \phi}{\partial z} + \frac{n_o}{c}\frac{\partial \phi}{\partial t} = \frac{2\pi}{n_o c\varepsilon}\left[\omega_o P_1 + 2\frac{\partial P_2}{\partial t}\right] . \tag{24}$$

This approximation, though somewhat arbitrary, leads to equations similar to those of McCall and Hahn which have been shown by experiment to be qualitatively correct (with regards to the behavior of $\varepsilon$). For this reason, its use is probably justified.

The preceding arguments indicate only that the $\frac{\partial P}{\partial t}$ terms in Eqs. (23) and (24) are not a priori negligible. In certain cases, they may, in fact, prove to be negligible. However, their neglect can only be justified on an a posteriori basis, if at all, by comparing the solutions obtained from Eqs. (23) and (24) with those obtained from Eqs. (11) and (20).

For the remainder of this paper, we will consider Eqs. (23) and (24) coupled with the Bloch equations

$$\frac{du}{dt} = -\left(\Delta\omega + \frac{\partial\phi}{\partial t}\right) v \ , \tag{25}$$

$$\frac{dv}{dt} = \left(\Delta\omega + \frac{\partial\phi}{\partial t}\right) u - \kappa\varepsilon w \ , \tag{26}$$

and

$$\frac{dw}{dt} = \kappa\varepsilon v \ . \tag{27}$$

Although Eqs. (23)-(27) have not been solved analytically and computer simulations have not been attempted, some information about the initial behavior of the phase modulation due to the resonance can be estimated by substituting the McCall and Hahn solutions into Eqs. (23) and (24). For these solutions, $P_1 = 0$, which implies $\frac{\partial P_1}{\partial t} = 0$, and Eq. (23) reduces to Eq. (11). As a result, we expect the initial behavior of the pulse envelope to be the same as that predicted by McCall and Hahn. As we will demonstrate later, this is indeed the case. Eq. (24), on the other hand, can be rewritten as

$$\frac{\partial\phi}{\partial z} + \frac{n_o}{c}\frac{\partial\phi}{\partial t} = \left(\frac{4\pi Np}{n_o c\varepsilon}\right)\frac{\partial}{\partial t}\,(\sin\,\delta)\int_{-\infty}^{\infty}\frac{g(\Delta\omega)d\Delta\omega}{1 + (\tau\Delta\omega)^2} \ . \tag{28}$$

Denoting the integral over $\Delta\omega$ by $\Gamma$ and carrying out the differentiation yields

$$\frac{\partial\phi}{\partial z} + \frac{n_o}{c}\frac{\partial\phi}{\partial t} = \left(\frac{4\pi Np\kappa}{n_o c}\right)\Gamma\,\cos\,4\left(\tan^{-1}(e^{\beta})\right) . \tag{29}$$

As the absorption coefficient $\alpha$ (the inverse Beer's length) is defined as $\alpha = 4\pi^2 Np\kappa\omega_o g(0)/n_o c$, Eq. (29) may be rewritten as

$$\frac{\partial\phi}{\partial z} + \frac{n_o}{c}\frac{\partial\phi}{\partial t} = \frac{\alpha}{\omega_o\pi g(0)}\,\Gamma\,\cos\left(4\,\tan^{-1}\,(e^{\beta})\right) . \tag{30}$$

Since $\Gamma > 0$ for any physically significant lineshape function, the phase equation has a non-zero driving term (except at the special points corresponding to the zeros of $\cos\left(4\tan^{-1}(e^{\beta})\right)$ and phase modulation will occur.

For the Lorentzian lineshape,

$$g(\Delta\omega) = \frac{g(0)}{1 + (T_2^*\Delta\omega)^2} , \tag{31}$$

where $g(0) = T_2^*/\pi$ and $T_2^*$ is the inhomogeneous contribution to the transverse relaxation time, $\Gamma$ may be evaluated explicitly,[15] with

$$\Gamma = \frac{\pi g(0)}{T_2^* + \tau} . \tag{32}$$

Thus,

$$\frac{\partial\phi}{\partial z} + \frac{n_o}{c}\frac{\partial\phi}{\partial t} = \frac{\alpha}{\omega_o(T_2^* + \tau)}\cos\left(4\tan^{-1}(e^{\beta})\right) . \tag{33}$$

Since $\cos\left(4\tan^{-1}(e^{\beta})\right)$ varies between +1 and -1, the magnitude of the phase driving term is $\alpha/\omega_o(T_2^* + \tau)$. Upon examination of Eq. (33), and by analogy to the area theorem,[4] it can be concluded that the characteristic distance $z_\phi$ associated with phase modulation effects is given by

$$z_\phi \approx \omega_o (T_2^* + \tau)/\alpha . \tag{34}$$

This is to be compared with the distance $z_\varepsilon$ over which the envelope will change appreciably. From the area theorem, we expect

$$z_\varepsilon \approx 1/\alpha . \tag{35}$$

Since $\omega_o T_2^*$ (or $\omega_o\tau$) is typically greater than $10^3$, $z_\phi$ is at least 3 orders of magnitude greater than $z_\varepsilon$ .

Figure 1a shows the hyperbolic secant pulse shape while Figure 1b shows the resulting phase driving function $\cos\left(4\tan^{-1}(e^{\beta})\right)$. Using the relation

$$\frac{\partial\phi}{\partial z} + \frac{n_o}{c}\frac{\partial\phi}{\partial t} = \left(\frac{n_oV-c}{cV\tau}\right)\frac{d\phi}{d\beta} \quad . \tag{36}$$

Eq. (33) can be integrated numerically. The resulting function $\phi(\beta)$ is plotted in Figure 1c. The constant $\gamma$ is given by

$$\gamma = \frac{\alpha cV\tau}{\omega_o(T_2^* + \tau)(n_oV-c)} \quad . \tag{37}$$

For the McCall and Hahn solution, the pulse velocity is given by

$$V \approx \frac{2(T_2^* + \tau)}{\alpha\tau^2} \ll c \tag{38}$$

and Eq. (37) reduces to

$$\gamma = -\frac{2}{\omega_o\tau} \quad . \tag{39}$$

The linear portion of $\phi$ corresponds to a constant frequency shift, $\delta\omega$, of magnitude

$$\delta\omega = \frac{\gamma}{\tau} = -\frac{2}{\omega_o\tau^2} \, . \tag{40}$$

Comparing $\delta\omega$ to the original bandwidth of the pulse, $d\omega = \frac{1}{\tau}$, we find

$$\frac{|\delta\omega|}{d\omega} = \frac{2}{\omega_o\tau} \ll 1 \, . \tag{41}$$

In the vicinity of the pulse peak, the center frequency chirps from $\omega_0 - \delta\omega$ to $\omega_0 + \delta\omega$ and back to $\omega_0 - \delta\omega$. However, since $|\delta\omega|$ is small compared to the bandwidth, this chirp will not be detectable. That $\delta\omega$ is not zero at $\beta = \pm\infty$ does not violate causality because the hyperbolic secant pulse itself is non-causal requiring an infinite amount of time to prepare.

As mentioned earlier, the characteristic distance over which significant deviations from the McCall and Hahn results should occur for a hyperbolic secant input pulse is $z_\phi$. From examination of Eqs. (25) and (26) with $\frac{\partial\phi}{\partial t}$ initially zero, it is readily apparent that v and u will be even and odd functions of $\Delta\omega$, respectively, for any un-phase-modulated input pulse. $P_1$ and $\frac{\partial P_1}{\partial t}$ are again initially zero, and although $\frac{\partial P_2}{\partial t}$ may differ from the McCall and Hahn result, it should still be of the same magnitude. As a consequence we may still expect that initially any un-phase-modulated input pulse will evolve toward the hyperbolic secant pulse while developing a negligibly small chirp. The characteristic distance associated with a breakdown of this behavior will again be $z_\phi$.

These predictions are in accord with the experimental results of Slusher and Gibbs.[6] In their experiment they propagated Hg laser pulses through Beer's lengths of Rb vapor. They observed considerable pulse reshaping but no phase modulation greater than 1/10 of the pulse bandwidth. Since $\omega_0\tau \approx 5\times10^6$ for their experiment, phase modulation should only become apparent after propagation through over a million Beer's lengths, if at all.

Even though the initial phase modulation is negligibly small, the $\frac{\partial\phi}{\partial t}$ terms in the Bloch equations will cause the polarizations, u and v, to deviate slightly from the McCall and Hahn solutions. $P_1$ and $\frac{\partial P_1}{\partial t}$ will no longer be zero, and $P_2$ and $\frac{\partial P_2}{\partial t}$ will no longer have the same behavior as derived earlier. These modified polarizations will then cause the envelope and phase to deviate from their initial behavior. This process may or may not lead to stable behavior. The question of stability will be addressed below.

The $2\pi$ hyperbolic secant solution of Eqs. (11)-(14) is known to be stable against large amplitude perturbations by virtue of the area theorem. It is also possible that stable solutions exist for Eqs. (23)-(27). Thus, it is instructive to attempt to derive an area theorem for this set of equations. Eq. (23) may be rewritten as

$$\frac{d\varepsilon}{dz} = \frac{2\pi N p \omega_0}{n_0 c} \int_{-\infty}^{\infty} \left[ v - \frac{2}{\omega_0} \frac{\partial u}{\partial t} \right] g(\Delta\omega)\, d\Delta\omega \; . \tag{42}$$

Defining the area $\theta$ by

$$\theta = \kappa \int_{-\infty}^{\infty} \varepsilon(t)\, dt \tag{43}$$

and using Eq. (25), Eq. (42) may be integrated to yield

$$\frac{d\theta}{dz} = \frac{-\alpha}{2\pi g(0)} \int_{-\infty}^{T} \int_{-\infty}^{\infty} \left[ \frac{1}{\Delta\omega + \frac{\partial\phi}{\partial t}} + \frac{2}{\omega_0} \right] \frac{\partial u}{\partial t}\, g(\Delta\omega)\, d\Delta\omega\, dt \; , \tag{44}$$

where $T \to \infty$. Using the initial conditions, $u(\Delta\omega) = v(\Delta\omega) = 0$ at $t = -\infty$, the second term in Eqs. (44) may be integrated over time yielding the equation

$$\frac{d\theta}{dz} = \frac{-\alpha}{2\pi g(0)} \int_{-\infty}^{T} \int_{-\infty}^{\infty} \frac{\frac{\partial u}{\partial t}}{\Delta\omega + \frac{\partial\phi}{\partial t}}\, g(\Delta\omega)\, d\Delta\omega\, dt$$
$$- \frac{\alpha}{\pi\omega_0 g(0)} \int_{-\infty}^{\infty} u(\Delta\omega, T)\, g(\Delta\omega)\, d\Delta\omega \; . \tag{45}$$

Integrating the first term in this expression by parts, we find

$$\frac{d\theta}{dz} = \frac{-\alpha}{2\pi g(0)} \int_{-\infty}^{\infty} \left[ \frac{1}{\Delta\omega + \frac{\partial\phi}{\partial t}(T)} + \frac{2}{\omega_o} \right] u(\Delta\omega,T)\, g(\Delta\omega)\, d\Delta\omega$$

$$- \frac{\alpha}{2\pi g(0)} \int_{-\infty}^{T} \int_{-\infty}^{\infty} \frac{u(\Delta\omega,t) \frac{\partial^2\phi}{\partial t^2}}{\left(\Delta\omega + \frac{\partial\phi}{\partial t}\right)^2} g(\Delta\omega)\, d\Delta\omega\, dt \tag{46}$$

If we assume that at some time $T_o$, the pulse has passed, so that $\varepsilon(t > T_o) = 0$ and $\frac{\partial\phi}{\partial t}(t > T_o) = \frac{\partial\phi_o}{\partial t}$ = const., we find from Eqs. (25) and (26) that

$$u(\Delta\omega,t) = u(\Delta\omega,T_o) \cos\left(\Delta\omega + \frac{\partial\phi_o}{\partial t}\right)(t - T_o)$$

$$+ v(\Delta\omega,T_o) \sin\left(\Delta\omega + \frac{\partial\phi_o}{\partial t}\right)(t - T_o) . \tag{47}$$

Substituting Eqs. (47) into Eq. (46), we note that as $t \to \infty$, the sinusoidal terms oscillate so rapidly that the principal contributions to the integrals come from $\Delta\omega + \frac{\partial\phi_o}{\partial t} = 0$. Thus

$$\frac{d\theta}{dz} = -\frac{\alpha}{2}\, v\left(-\frac{\partial\phi_o}{\partial t}, T_o\right) - \frac{\alpha}{\omega_o T_2}\, u\left(-\frac{\partial\phi_o}{\partial t}, T_o\right)$$

$$- \frac{\alpha}{2\pi g(0)} \int_{-\infty}^{T} \int_{-\infty}^{\infty} \frac{u(\Delta\omega,t) \frac{\partial^2\phi}{\partial t^2}}{\left(\Delta\omega + \frac{\partial\phi}{\partial t}\right)^2} g(\Delta\omega)\, d\Delta\omega\, dt . \tag{48}$$

As the integrand of the third term in Eq. (48) is an extremely complex function of time, the integral may not be explicitly evaluated. However, as long as $\frac{\partial\phi}{\partial t}$ is sufficiently small, this term is negligible. Similarly, in the limit as $\frac{\partial\phi_o}{\partial t}$ goes to zero, it is evident that u and v approach the McCall and Hahn results,

$$v(0,T_o) = \sin\theta \tag{49a}$$

and

$$u(0,T_0) = 0 \ , \tag{49b}$$

and we obtain the area theorem

$$\frac{d\theta}{dz} = -\frac{\alpha}{2}\sin\theta . \tag{50}$$

For small but finite $\frac{\partial\phi}{\partial t}$, no simple relations for u and v can be obtained. However, it is reasonable to expect that the behavior of the polarizations will not differ significantly from the McCall and Hahn results (and Eq. (50) will be approximately valid) until $\frac{\partial\phi}{\partial t}$ becomes comparable to the pulse bandwidth. This will not occur (if it occurs at all) until the pulse has travelled a distance of the order of $\frac{\omega_o T_2^*}{\alpha}$ . If $\frac{\partial\phi}{\partial t}$ becomes large, then the integral term in Eq. (48) should also contribute.

The general questions of whether $\frac{\partial\phi}{\partial t}$ will grow without bound and whether or not stable solutions exist have not been answered by the preceding qualitative analysis. As this system of equations is not amenable to normal nonlinear stability analysis, these questions must be answered by computer simulation. The numerical solution of Eqs. (23)-(27) for large propagation distances should prove very interesting.

In conclusion, we have shown that phase modulation will occur in pure self-induced transparency contrary to the results of most previous authors. The initial phase modulation is much less than the pulse bandwidth and may only become appreciable after propagation through thousands of Beer's lengths. As this phase modulation will modify the propagation characteristics of the envelope function, the question of whether or not stable solutions exist has been raised. Provided the phase modulation is small, it has been shown that the area theorem of McCall and Hahn is still approximately valid. If $\frac{\partial\phi}{\partial t}$ grows without bound, the stability of the $2\pi$ hyperbolic secant pulse will break down after propagation through a distance comparable to $\frac{\omega_o T_2^*}{\alpha}$ . The question of stability must be left to numerical simulation. The main conclusion of this work is that although phase modulation is present and the stability of the $2\pi$ hyperbolic secant may eventually break down, the McCall and Hahn results are valid to a high degree of approximation in nearly all cases of experimental interest. This justifies the neglect of the $\frac{\partial P}{\partial t}$ terms in most situations. However, the possible application of self-induced transparency to optical computers or atmospheric propagation may require their retention in simulation calculations.

## APPENDIX

The wave equation for an electric field propagating in a nonlinear medium of refractive index $n_o$ may be written as

$$\frac{\partial^2 E_+}{\partial z^2} - \left(\frac{n_o}{c}\right)^2 \frac{\partial^2 E_+}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2 P_+}{\partial t^2} \quad , \tag{A1}$$

where $E_+(z,t)$ is the electric field and $P_+(z,t)$ is the nonlinear polarization. This equation is difficult to handle mathematically. As a result, it is useful to find an approximate expression(s) for Eq. (A1) which is easier to use. Assume solutions of the form

$$E_+(z,t) = \varepsilon(z,t)\ A \tag{A2}$$

and

$$P_+(z,t) = \Big(P_1(z,t) + iP_2(z,t)\Big)\ A, \tag{A3}$$

where

$$A = e^{i[\omega_o t - k_o z - \phi(z,t)]} \quad . \tag{A4}$$

The envelope of the electric field, $\varepsilon(z,t)$, is forced to remain real while no such restriction is placed on the envelope of the polarization. The phase modulation function, $\phi(z,t)$, is left arbitrary.

Using the results

$$\frac{\partial A}{\partial z} = -\ i\left(k_o + \frac{\partial \phi}{\partial z}\right) A \tag{A5}$$

and

$$\frac{\partial A}{\partial t} = i\left(\omega_o - \frac{\partial \phi}{\partial t}\right) A \ , \tag{A6}$$

the derivatives in Eq. (A1) may be evaluated. Thus differentiating $E_+$ with respect to z yields

$$\frac{\partial E_+}{\partial z} = \left[\frac{\partial \varepsilon}{\partial z} - i\varepsilon\left(k_o + \frac{\partial \phi}{\partial z}\right)\right] A \quad . \tag{A7}$$

A second differentiation with respect to z yields

$$\begin{aligned}\frac{\partial^2 E_+}{\partial z^2} &= \left[\frac{\partial^2 \varepsilon}{\partial z^2} - i\left(k_o + \frac{\partial \phi}{\partial z}\right)\frac{\partial \varepsilon}{\partial z} - i\varepsilon\,\frac{\partial^2 \phi}{\partial z^2}\right] A - i\left(k_o + \frac{\partial \phi}{\partial z}\right)\left[\frac{\partial \varepsilon}{\partial z} - i\left(k_o + \frac{\partial \phi}{\partial z}\right)\varepsilon\right] A \\ &= \left[\frac{\partial^2 \varepsilon}{\partial z^2} - \left(k_o + \frac{\partial \phi}{\partial z}\right)^2 \varepsilon\right] A - i\left[2\left(k_o + \frac{\partial \phi}{\partial z}\right)\frac{\partial \varepsilon}{\partial z} + \varepsilon\,\frac{\partial^2 \phi}{\partial z^2}\right] A \quad .\end{aligned} \tag{A8}$$

Similar differentiations of $E_+$ with respect to t yield

$$\frac{\partial E_+}{\partial t} = \left[\frac{\partial \varepsilon}{\partial t} + i\varepsilon\left(\omega_o - \frac{\partial \phi}{\partial t}\right)\right] A \tag{A9}$$

and

$$\begin{aligned}\frac{\partial^2 E_+}{\partial t^2} &= \left[\frac{\partial^2 \varepsilon}{\partial t^2} + i\left(\omega_o - \frac{\partial \phi}{\partial t}\right)\frac{\partial \varepsilon}{\partial t} - i\varepsilon\,\frac{\partial^2 \phi}{\partial t^2}\right] A + i\left(\omega_o - \frac{\partial \phi}{\partial t}\right)\left[\frac{\partial \varepsilon}{\partial t} + i\varepsilon\left(\omega_o - \frac{\partial \phi}{\partial t}\right)\right] A \\ &= \left[\frac{\partial^2 \varepsilon}{\partial t^2} - \left(\omega_o - \frac{\partial \phi}{\partial t}\right)^2 \varepsilon\right] A + i\left[2\left(\omega_o - \frac{\partial \phi}{\partial t}\right)\frac{\partial \varepsilon}{\partial t} - \varepsilon\,\frac{\partial^2 \phi}{\partial t^2}\right] A \quad .\end{aligned} \tag{A10}$$

Likewise, two differentiations of $P_+$ with respect to t yield

$$\frac{\partial P_+}{\partial t} = \left[\frac{\partial P_1}{\partial t} + i\frac{\partial P_2}{\partial t} + i\left(\omega_o - \frac{\partial \phi}{\partial t}\right)\left(P_1 + P_2\right)\right] A \tag{A11}$$

and

$$\frac{\partial^2 P_+}{\partial t^2} = \left[\frac{\partial^2 P_1}{\partial t^2} + i\,\frac{\partial^2 P_2}{\partial t^2} - i\left(P_1 + iP_2\right)\frac{\partial^2\phi}{\partial t^2} + i\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\left(\frac{\partial P_1}{\partial t} + i\,\frac{\partial P_2}{\partial t}\right)\right]A$$

$$+\; i\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\left[\frac{\partial P_1}{\partial t} + i\,\frac{\partial P_2}{\partial t} + i\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\left(P_1 + P_2\right)\right]A$$

$$\text{(A12)}$$

$$= \left[\frac{\partial^2 P_1}{\partial t^2} + P_2\,\frac{\partial^2\phi}{\partial t^2} - 2\left(\omega_o\,\frac{\partial\phi}{\partial t}\right)\frac{\partial P_2}{\partial t} - \left(\omega_o - \frac{\partial\phi}{\partial t}\right)^2 P_1\right] A$$

$$+\; i\left[\frac{\partial^2 P_2}{\partial t^2} - P_1\,\frac{\partial^2\phi}{\partial t^2} + 2\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\frac{\partial P_1}{\partial t} - \left(\omega_o - \frac{\partial\phi}{\partial t}\right)^2 P_2\right]A\,.$$

After substituting the results of Eqs. (A8), (A10), and (A12) into Eq. (A1), equating the real and imaginary parts yields the two equations

$$\left[\frac{\partial^2\varepsilon}{\partial z^2} - \left(k_o + \frac{\partial\phi}{\partial z}\right)^2\varepsilon\right] - \left(\frac{n_o}{c}\right)^2\left[\frac{\partial^2\varepsilon}{\partial t^2} - \left(\omega_o - \frac{\partial\phi}{\partial t}\right)^2\varepsilon\right.$$

$$= \frac{4\pi}{c^2}\left[\frac{\partial^2 P_1}{\partial t^2} + P_2\,\frac{\partial^2\phi}{\partial t^2} - 2\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\frac{\partial P_2}{\partial t} - \left(\omega_o - \frac{\partial\phi}{\partial t}\right)^2 P_1\right] \qquad \text{(A13)}$$

and

$$\left[2\left(k_o + \frac{\partial\phi}{\partial z}\right)\frac{\partial\varepsilon}{\partial z} + \varepsilon\,\frac{\partial^2\phi}{\partial z^2}\right] - \left(\frac{n_o}{c}\right)^2\left[-\,2\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\frac{\partial\varepsilon}{\partial t} + \varepsilon\,\frac{\partial^2\phi}{\partial t^2}\right]$$

$$= \frac{4\pi}{c^2}\left[-\,\frac{\partial^2 P_2}{\partial t^2} + P_1\,\frac{\partial^2\phi}{\partial t^2} - 2\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\frac{\partial P_1}{\partial t} + \left(\omega_o - \frac{\partial\phi}{\partial t}\right)^2 P_2\right]. \qquad \text{(A14)}$$

Since $\omega_o = ck_o/n_o$, Eqs. (A13) and (A14) may be rewritten as

$$\left[\frac{\partial^2\varepsilon}{\partial z^2} - \left(\frac{n_o}{c}\right)^2\frac{\partial^2\varepsilon}{\partial t^2}\right] - 2k_o\varepsilon\left[\frac{\partial\phi}{\partial z} + \left(\frac{n_o}{c}\right)\frac{\partial\phi}{\partial t}\right] - \varepsilon\left[\left(\frac{\partial\phi}{\partial z}\right)^2 - \left(\frac{n_o}{c}\right)^2\left(\frac{\partial\phi}{\partial t}\right)^2\right.$$

$$= \frac{4\pi}{c^2}\left[\frac{\partial^2 P_1}{\partial t^2} + P_2\,\frac{\partial^2\phi}{\partial t^2} - 2\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\frac{\partial P_2}{\partial t} - \left(\omega_o - \frac{\partial\phi}{\partial t}\right)^2 P_1\right] \qquad \text{(A15)}$$

and

$$\varepsilon\left[\frac{\partial^2\phi}{\partial z^2} - \left(\frac{n_o}{c}\right)^2 \frac{\partial^2\phi}{\partial t^2}\right] + 2k_o\left[\frac{\partial\varepsilon}{\partial z} + \left(\frac{n_o}{c}\right)\frac{\partial\varepsilon}{\partial t}\right] + \left[2\ \frac{\partial\phi}{\partial z}\frac{\partial\varepsilon}{\partial z} - \left(\frac{n_o}{c}\right)^2 \frac{\partial\phi}{\partial t}\frac{\partial\varepsilon}{\partial t}\right]$$

$$= \frac{4\pi}{c^2}\left[-\frac{\partial^2 P_2}{\partial t^2} + P_1\frac{\partial^2\phi}{\partial t^2} - 2\left(\omega_o - \frac{\partial\phi}{\partial t}\right)\frac{\partial P_1}{\partial t} + \left(\omega_o - \frac{\partial\phi}{\partial t}\right)^2 P_2\right]. \tag{A16}$$

Equations (A15) and (A16) are still exact. Now we make the slowly-varying envelope approximation. That is, we assume that neither the envelope of the electric field nor the envelope of the polarization will change significantly over one wavelength or during one optical cycle. In this approximation

$$\omega_o^2\ |\varepsilon| \gg \omega_o \left|\frac{\partial\varepsilon}{\partial t}\right| \gg \left|\frac{\partial^2\varepsilon}{\partial t^2}\right| \quad , \tag{A17a}$$

$$k_o^2\ |\varepsilon| \gg k_o \left|\frac{\partial\varepsilon}{\partial z}\right| \gg \left|\frac{\partial^2\varepsilon}{\partial z^2}\right| \quad , \tag{A17b}$$

$$\omega_o^2\ |P_{1,2}| \gg \omega_o \left|\frac{\partial P_{1,2}}{\partial t}\right| \gg \left|\frac{\partial^2 P_{1,2}}{\partial z^2}\right| \quad , \tag{A17c}$$

and

$$k_o^2\ |P_{1,2}| \gg k_o \left|\frac{\partial P_{1,2}}{\partial z}\right| \gg \left|\frac{\partial^2 P_{1,2}}{\partial z^2}\right| \quad . \tag{A17d}$$

Similarly, we will make the slowly varying phase approximation where

$$\omega_o \gg \frac{\partial\phi}{\partial t} \tag{A18a}$$

and

$$k_o \gg \frac{\partial\phi}{\partial z} \quad . \tag{A18b}$$

Using these two approximations, Eqs. (A15) and (A16) reduce to

$$\left[\frac{\partial^2 \varepsilon}{\partial z^2} - \left(\frac{n_o}{c}\right)^2 \frac{\partial^2 \varepsilon}{\partial t^2}\right] - 2k_o \varepsilon \left[\frac{\partial \phi}{\partial z} + \left(\frac{n_o}{c}\right) \frac{\partial \phi}{\partial t}\right]$$

$$= \frac{4\pi}{c^2} \left[P_2 \frac{\partial^2 \phi}{\partial t^2} - 2\omega_o \frac{\partial P_2}{\partial t} - \omega_o^2 P_1\right] \tag{A19}$$

and

$$\varepsilon \left[\frac{\partial^2 \phi}{\partial z^2} - \left(\frac{n_o}{c}\right)^2 \frac{\partial^2 \phi}{\partial t^2}\right] + 2k_o \left[\frac{\partial \varepsilon}{\partial z} + \left(\frac{n_o}{c}\right) \frac{\partial \varepsilon}{\partial t}\right]$$

$$= \frac{4\pi}{c^2} \left[P_1 \frac{\partial^2 \phi}{\partial t^2} - 2\omega_o \frac{\partial P_1}{\partial t} + \omega_o^2 P_2\right] \quad . \tag{A20}$$

These two equations represent the greatest simplification of Eqs. (A15) and (A16) consistent with only the slowly varying envelope and phase approximations. Further simplification requires either more approximations or a posteriori knowledge of the relative importance of different terms.

## REFERENCES

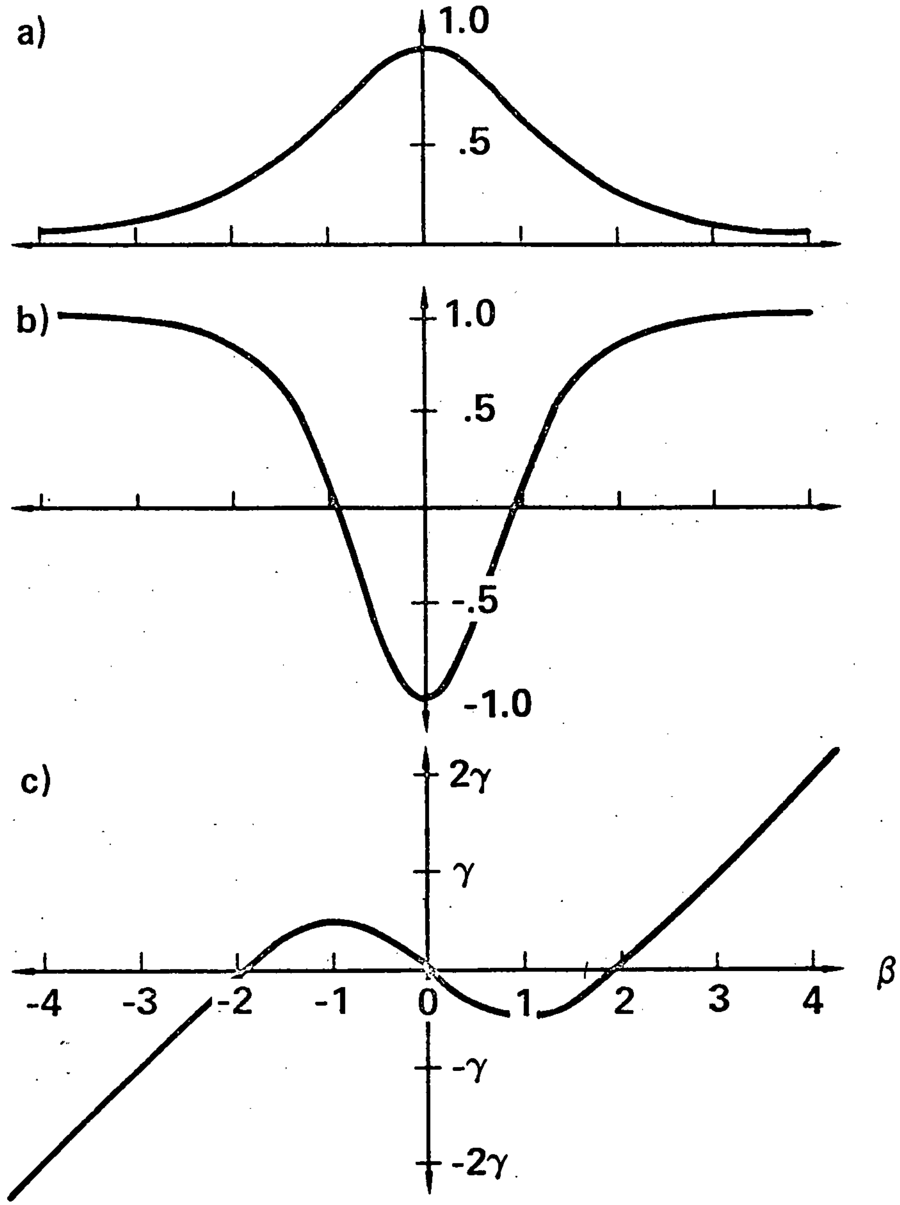
1. S. L. McCall, Ph.D. Thesis, U.C. Berkeley, 1968 (unpublished).
2. The equations in McCall's thesis were modified to yield Eqs. (6)-(8) by substituting $\Delta\omega + \frac{\partial\phi}{\partial t}$ for $\Delta\omega$. This transforms his equations in a frame rotating with angular frequency $\omega_0$ to those in a frame rotating with angular frequency $\omega_0 - \frac{\partial\phi}{\partial t}$.
3. S. L. McCall and E. L. Hahn, Phys. Rev. Letts., 18, 908 (1967).
4. S. L. McCall and E. L. Hahn, Phys. Rev., 183, 457 (1969).
5. A. Icsevgi and W. E. Lamb, Phys. Rev., 185, 517 (1969).
6. R. E. Slusher and H. M. Gibbs, Phys. Rev., 5A, 1634 (1972).
7. G. L. Lamb, Rev. Mod. Phys., 43, 99 (1971)
8. L. Matulic and J. H. Eberly, Phys. Rev., 6A, 822 (1972).
9. L. Matulic, Opt. Comm., 2, 249 (1970).
10. J. C. Diels and E. L. Hahn, Phys. Rev., 8A, 1084 (1973).
11. C. T. Lee, Opt. Comm., 9, 1 (1973).
12. L. Davidovich and J. H. Eberly, Opt. Comm., 3, 32 (1971).
13. J. H. Eberly and L. Matulic, Opt. Comm., 1, 241 (1969).
14. R. C. Harney, unpublished work.
15. G. Petit Bois, Tables of Indefinite Integrals (Dover Publications, New York, 1961), p. 17.

## FIGURE CAPTION

Figure 1. a) the sech $\beta$ pulse shape of McCall and Hahn.

b) plot of the function $\cos\left(4 \tan^{-1} (e^{\beta})\right)$, which is proportional to $\frac{d\phi}{dt}$ .

c) plot of $\phi(\beta)$ obtained by numerically integrating Eq. (33) subject to the condition $\phi(\beta=0) = 0$.


a)
1.0
.5
b)
1.0
.5
-.5
-1.0
c)
2γ
γ
-γ
-2γ
β
-4
-3
-2
-1
0
1
2
3
4

# DISTRIBUTION

## Internal

| | |
|---|---|
| F. Aldridge | L-500 |
| J. Anderson | L-313 |
| R. Batzel | L-1 |
| A. Bernhardt | L-500 |
| W. Bischel | L-213 |
| S. Bloom | L-313 |
| R. Borg | L-231 |
| T. Burgess | L-156 |
| L. Coleman | L-551 |
| J. Emmett | L-555 |
| A. Glass | L-547 |
| P. Hoff | L-213 |
| W. Hoover | L-504 |
| R. Jacobs | L-213 |
| B. Johnson | L-547 |
| G. Kachen | L-547 |
| N. Keeler | L-71 |
| W. Krupke | L-555 |
| C. Layne | L-547 |
| G. Leppelmeier | L-541 |
| H. Lowdermilk | L-547 |
| J. Marling | L-500 |
| F. Milanovich | L-613 |
| M. Moran | L-503 |
| C. Rhodes | L-213 |
| M. Schwab | L-503 |
| E. Teller | L-0 |
| T. Wainwright | L-71 |
| M. Weber | L-547 |
| L. Wood | L-0 |
| R. Harney (30) | L-613 |
| Dept. of App. Sci. (5) | L-699 |
| TID Library (5) | L-9 |

## External

Y. Yeh
F. Wooten
Department of Applied Science
U.C. Davis
Davis, California